Warwick Mathematics Exchange

PX148

Classical Mechanics and Special Relativity

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RisingStar111, aka The One That Plays Osu!

Contents

Introduction

Classical Mechanics covers the basics behind equations of motion and energy and utilises many useful techniques for simplifying problems. Special Relativity covers the even more basics behind equations of motion at high speeds and the unintuitive nature of the results.

Disclaimer: This document was made by a first year student. I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. Additionally, this document was written at the end of the 2022 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be <u>underlined</u>. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems. It also hasn't been included :)

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

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Authors

This document was written by A.K.Knight, a mathphys student, mostly following the 2022 lecture notes. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @RisingStar111#0563 or send a message in the related channels in teh WMX or Physics discord servers for any corrections/additions/layout changes. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support some random person on the maths course whom I stole this document's format from (not me), you can buy them a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

1 Classical Mechanics

1.1 Newton's Laws

Newton's First Law - Bodies remain at constant velocity unless acted upon by a (net) force.

Newton's Second Law - Defining momentum as $p = mv$, Force is the rate of change of momentum, so

$$\vec{F} = \frac{dp}{dt} = m\frac{dv}{dt} = m\vec{a} \tag{1}$$

Newton's Third Law - Every action (force) has an equal and opposite reaction (force).

1.2 Forces

Scalars - Quantities like mass and speed whose values are defined only by their magnitude (can be negative).

Vectors - Quantities like force and velocity whose values are defined by both their magnitude and direction. Arrows should be used to indicate vectors, as used in this document.

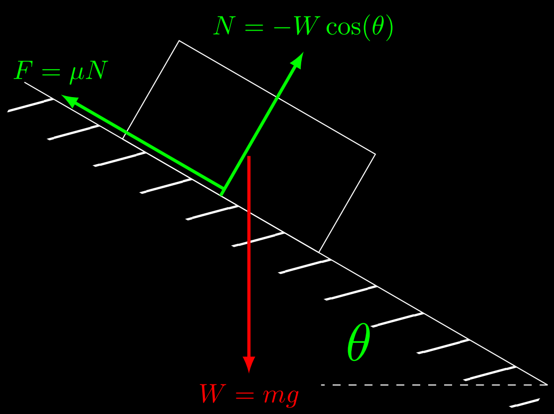
Normal reaction force - Part of the contact force, perpendicular to the surface.

Friction force - (The other) part of the reaction force, parallel to the surface.

$$F \leq \mu_S N \text{ and } F = \mu_K N \tag{2}$$

using the coefficient of static μ_S or kinetic μ_K friction. In general, $\mu_K < \mu_S$

Example of one of those diagrams you should be drawing - typical block on slope setup.



Force of Gravity - The attractive force from an object with mass m_1 on an object with mass m_2 given by

$$F = \frac{Gm_1m_2}{r^2} \quad \text{or in vector form} \quad \vec{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{r} \tag{3}$$

Newton's shell theorem - The force from a uniform spherical shell/sphere on a particle is the same as the force from a particle with the same mass as the sphere/[mass of sphere contained within radius to particle] at the origin.

Forces on systems of particles - Total external force affects the centre of mass of the system.

$$\sum \vec{F} = m_{total} a_{cm} \tag{4}$$

Centre of mass - For distinct bodies on left, and continous bodies with density function on right, the centre of mass (cm) of the system is (note: $dm = f(\rho)dV$)

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_{cm_i}}{\sum m_i} \qquad \vec{r}_{cm} = \frac{\int \vec{r}\, dm}{\int dm} \tag{5}$$

1.3 Properties of Objects

Lots of silly words to remember :⊃

Smooth - No friction.
Light - No mass.
Massive - Has mass. Do not assume that it is large/has size.
Rigid - Can not bend.
Flexible - Just think of a string.
Ideal - Codeword for applying a bunch of these words. Ideal strings are light flexible not stretchy, ideal pulleys are light frictionless rigid, etc.
Uniform - Constant density.
Elastic - No (kinetic) energy is lost on collision.
Particle - All mass is located at a single point (no size/rotation).

Pulleys - Redirect forces through strings. Tension in the string is constant if the pulley and string are ideal.

1.4 Acceleration

Constant acceleration - Can use SUVAT ($v = u + at$, $s = ut + \frac{1}{2}at^2$, $s = vt - \frac{1}{2}at^2$, $v^2 = u^2 + 2as$, $s = \frac{v+u}{2}t$)

Time-dependant acceleration - Same process as deriving SUVAT but a is a function of time t.

Position-dependant acceleration - a is a function of position x. Chain rule leads to

$$\frac{1}{2}(v^2 - u^2) = \int_0^x a\,dx \tag{6}$$

Velocity-dependant acceleration - Solve and rearrange the following for $v(t)$ and integrate for $x(t)$.

$$\int_u^v \frac{dv}{a(v)} = \int_0^t dt = t \tag{7}$$

2 Work and Energy

Kinetic Energy - Defined as $T = \frac{1}{2}mv^2$.

Work Done - $W = \int F\,dx = \int dT = \Delta T$ i.e Work Done (on an object) is the Change in Kinetic Energy. (For general $\vec{F}$, $dW = \vec{F} \cdot d\vec{r} = F\cos(\theta)\,dr = dT$)

Conservative Field - Work done is independant of path travelled, resulting in loops in a conservative field doing no net work.

Potential Energy - In a conservative field, the energy associated with a position is called $U = -W$(from a fixed point O to an arbitrary point P). Negative so that work from high to low potential is positive

Total Energy - Total energy E, for a conservative field, $E = T + U =$ constant.

Forces from Potential - $\vec{F} = -\nabla U$.

Be very careful with signs when using Work - When *you* do work on an object, $W = \Delta U$ i.e you do work to move things to higher potentials, while the *Force* (of the field) does work to move objects to lower potential.

Gravitational Potential Energy - $U(r) = -\frac{GMm}{r}$. (If deriving, usually $U(\infty) = 0$). 'Gravitational potential' $\phi(r)$ can sometimes be used to mean potential per unit mass, $u(r) = -\frac{GM}{r}$.

Power - $P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$

2.1 Rockets

Rockets? - Remember to take into account the change in mass $dm < 0$ as the rocket burns fuel (leaving at speed v_e from the rocket) and conserve momentum. (Ik it's weird, but after time dt, m goes to $m + dm$ so $dm < 0$)

$$m\,dv = -v_e\,dm \tag{8}$$

The Rocket Equation - From integrating ↑, with m_0, v_0 the initial mass/speed and m, v the final mass/speed.

$$v - v_0 = \Delta v = v_e \ln\left(\frac{m_0}{m}\right) \tag{9}$$

Thrust - Because we only care about the lowest order infinitesimals, can write

$$\text{change in momentum of the Rocket(plus fuel) } dp = -v_e \, dm \tag{10}$$

$$F = \frac{dp}{dt} = -v_e \frac{dm}{dt}, \qquad dm < 0 \text{ remember} \tag{11}$$

2.2 Collisions

Generalised Collisions - Momentum is conserved so so is the speed of the centre of mass $v_c m$. Potentials don't matter as interactions are short range and over short times.
You can resolve multidimensional collisions into components as expected.

$$\vec{v}_{cm} = \frac{\vec{p}_{cm}}{m_{cm}} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \tag{12}$$

Inelastic Collisions - Particles stick together, so $\vec{v}_1 = \vec{v}_2 = \vec{v}_{cm}$

Elastic Collisions - Particles' speed in the CoM frame stay constant (note *speed* not *velocity*).

3 Simple Harmonic Motion (SHM)

The Equation - $\ddot{x} + \omega^2 x = 0$

General Solution to SHM - $x = A\cos(\omega t) + Bsin(\omega t) = C\cos(\omega t + \phi) = De^{i\omega t}$ where A,B,C are real constants, and D is a complex constant $Ce^{i\phi}$. When using complex form, physical results are obtained with the real part of the solution.

Frequency - $f = \frac{\omega}{2\pi} = \frac{1}{T}$ where T is the 'Time Period'.

3.1 Springs

Mass on a Spring - Intuition and derivation from $F = -kx$

$$\frac{d^2x}{dt^2} = -\omega^2 x \text{ where } \omega^2 = \frac{k}{m} \tag{13}$$

Potential of a Spring - Using the work *you* do *against* the spring force and defining x=0 as zero potential,

$$W = \int_0^x F\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2 = U \tag{14}$$

Note: Parabolic potential $U \propto x^2$ usually means SHM.

Average energy of a Spring - $\langle T\rangle = \langle U\rangle = \frac{1}{4}kA^2$

Pendulum - Small angle approximations ftw

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) \tag{15}$$

$$\omega^2 = \frac{g}{l} \tag{16}$$

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{17}$$

3.2 Damping

Damped SHM - $\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$ where $\gamma = \frac{b}{m}$ where b is from the damping force $-b\dot{x}$.

Solution to Damped SHM - Using the complex form,

$$\text{Solution guess of } z = Ae^{pt} \text{ leads to} \tag{18}$$

$$p = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega^2} \tag{19}$$

Modes of Damping - Depending on p, the physical motion described by the damping equation will change:

$\gamma = 0$ - No damping occurs (there's no resistive force).

$\gamma < 2\omega$ - light/under-damping (still oscillates, but magnitude decreases)

$$x = |a|e^{-\gamma t/2}\cos(\omega' t + \phi), \text{ where } \omega' = \sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2} \tag{20}$$

$\gamma > 2\omega$ - heavy/over-damping (no oscillation, just returns to rest)

$$z = Ae^{p_1 t} + Be^{p_2 t}, \quad \text{(remember } x = \Re(z)) \tag{21}$$

where

$$p_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega^2} \tag{22}$$

$\gamma = 2\omega$ - critical damping (fastest return to rest)

$$z = (a + bt)e^{-\gamma t/2} \tag{23}$$

3.3 Forced Oscillation

Forced Oscillation - A force $F = F_0\cos(\omega t)$ is applied to the damped system, giving

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m}\cos(i\omega t) \tag{24}$$

ω_0 is the natural frequency, as opposed to the applied frequency ω

$$z = |a|e^{i(\omega t + \phi)} \text{ so } x = \Re(z) = |a|\cos(\omega t + \phi) \tag{25}$$

$$a = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega} \tag{26}$$

$$|a| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \tag{27}$$

$$\tan(\phi) = -\frac{\gamma\omega}{\omega_0^2 - \omega^2} \tag{28}$$

$$\frac{|a|}{|a|_0} = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \tag{29}$$

no you can't tell me what to do

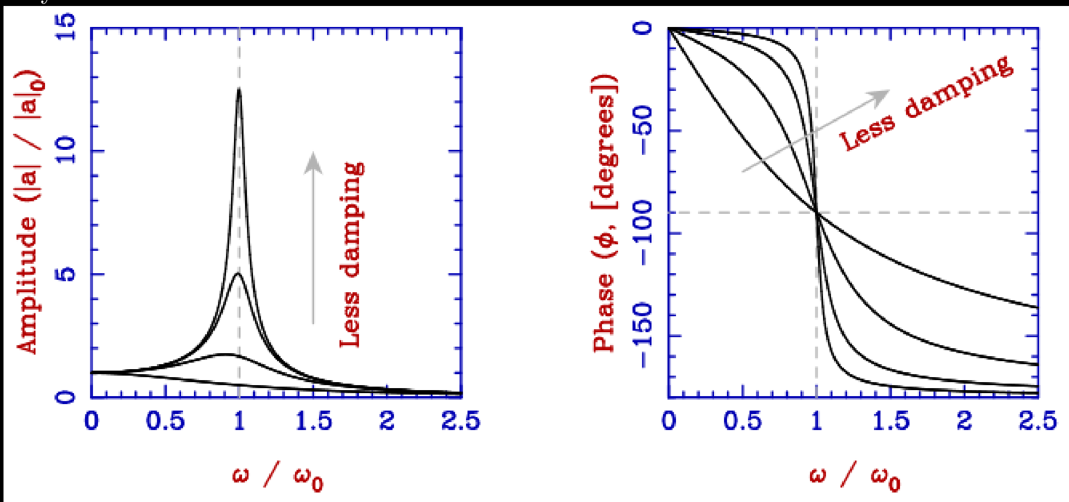


Figure: *Left:* amplitude versus driving frequency. *Right:* phase angle ϕ versus frequency. A negative angle means the displacement lags behind the driving force.

Resonance - For lightly damped systems, as ω approaches ω_0 the amplitude increases, peaking when they're equal

Phase difference - The system gets increasingly out of phase, lagging behind the applied force, as ω increases. $\phi = 0$ for $\omega \ll \omega_0$, $\phi = -90°$ for $\omega = \omega_0$ and $\phi \to -180°$ as $\omega \to \infty$

Other fun resonance stuff - The 'Full Width at Half Maximum' FWHM plotted against energy ($\propto |a|^2$) is $\approx \gamma$
The peak $\frac{|a|}{|a|_0} \approx \frac{\omega_0}{\gamma} = Q$ the 'Q-factor'.

4 Circular Motion

Angular Velocity - Defining the angular velocity $\omega = \frac{d\theta}{dt}$

$$v = \omega r \tag{30}$$

$$a = \omega v = \omega^2 r = \frac{v^2}{r} \tag{31}$$

$$Period = \frac{2\pi}{\omega} \tag{32}$$

Or as as vectors so that direction of rotation is defined, with $\vec{\omega}$ along the axis of rotation (order is important here)

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{33}$$

Circular Orbits - Balancing force from gravity and force needed to maintain an angular velocity

$$F = \frac{GMm}{R^2} = m\omega^2 R = ma \tag{34}$$

$$\omega^2 = \frac{GM}{R^3} = \frac{4\pi^2}{P^2} \tag{35}$$

Kepler's Third Law - $P^2 \propto R^3$

4.1 Offset Forces

Moments - 'Rotational force' given by Fx with $\vec{F}$ perpendicular to $\vec{x}$

Torques - Generalised Moments, measured with respect to a chosen Origin (if this point changes, so does the torque)

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{36}$$

Defining the angular momentum $\vec{L} = \vec{r} \times \vec{p}$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{37}$$

The torque is the rate of change of momentum

Systems - Total external torque equals the rate of change of total angular momentum.

Equilibrium - A body in equilibrium will have zero torque around any point.

Centre of gravity - The point about which gravitational forces have 0 total torque (likely not the centre of gravity unless the gravitational field is constant, in which case they are always the same).

4.2 Angular Momentum

Angular Momentum on a Circle - When the origin (for calculating torques) is at the centre of rotation for circular motion,

$$L = mrv = mr^2\omega \text{ parallel to and in the same direction as the axis of rotation } \vec{\omega} \tag{38}$$

Moment of Inertia - $I = mr^2$

Kinetic Energy - In this special case, $\vec{L} = I\vec{\omega}$ Second

$$T = \frac{1}{2}I\omega^2 \tag{39}$$

Rigid Bodies - If all parts of a body are rotating around the same axis but not necessarily in the same plane, defining

$$I = \sum m_i r_{i\,i}^2 = \int r^2 \, dm = \int r^2 \rho \, dV \tag{40}$$

with r the perpendicular distance to the axis of rotation at a point $\vec{r}$, we keep the relation $T = \frac{1}{2}I\omega^2$. Additionally, in this module, $\vec{L} = I\vec{\omega}$ always holds.

Angular Acceleration - As (in this module) $\vec{\tau}$ is parallel to $\vec{\omega}$, then angular acceleration $\frac{d\omega}{dt}$ is defined in

$$\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2} \tag{41}$$

4.3 Using Moment of Inertia

General Problems - Take care over the position at which forces act and remember torque exists.

Parallel Axis Theorem - The moment of inertia around an axis a perpendicular distance d away from an object's centre of mass is the same as the moment of inertia around a parallel axis through the objects centre of mass plus Md^2

Conservation of Angular Momentum - Under no net external torques, angular momentum is conserved.

General Kinetic Energy - When applying a force to an unconstrained object, offset from its centre of mass, both a torque and acceleration are created

$$T = \frac{1}{2}mv_cm^2 + \frac{1}{2}I\omega^2 \tag{42}$$

with I measured around an axis through the centre of mass and parallel to $\vec{\omega}$

Angular Power - $dW = \tau d\theta$ so $P = \vec{\tau} \cdot \vec{\omega}$

$$I = I(\|) + Md^2 \tag{43}$$

Useful Moments of Inertia - May be asked to derive these, but nice to know these for other things too
Ring about an axis through its centre - MR^2
Uniform disk about an axis through its centre - $\frac{1}{2}MR^2$
Uniform sphere about an axis through its centre - $\frac{2}{5}MR^2$
Thin rod length L, about a perpendicular axis through its centre - $\frac{1}{12}ML^2$
Thin rod length L, about a perpendicular axis through an end - $\frac{1}{3}ML^2$

5 Special Relativity

5.1 Time and Position

Rules of Relativity - Physics is the same in all reference frames. The speed of light is constant.

The Lorentz Transforms - Remember these. To go backwards sub $-u$ for u. Other dimensions are constant if the frames are defined nicely.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{44}$$

$$t' = \gamma(t - ux/c^2) \tag{45}$$

$$x' = \gamma(x - ut) \tag{46}$$

Length Contraction - $L_u = L_0/\gamma$, L_u is the length measured (Simultaneously at 2 locations) in the frame where the object is moving at speed u.

Time Dilation - $\Delta T_u = \gamma \Delta T_0$, ΔT_u is the change in time measured (between two events) in the frame where the object is moving at speed u.

5.2 Velocity

Doppler Effect - + for moving away, - for moving towards

$$\frac{f_0}{f} = \frac{\lambda}{\lambda_0} = \gamma\left(1 \pm \frac{u}{c}\right) \tag{47}$$

Addition of Velocities - u is the speed you are travelling relative to a stationary observer in frame S', v' is the speed the other thing is moving to the same stationary observer, v is the speed you see the other thing moving at in your frame S. For the inverse, sub $-u$ for u and swap v' and v

$$v = \frac{v' + u}{1 + uv'/c^2} \tag{48}$$

Additionally for multiple dimensions

$$v_{y,z} = \frac{v'_{y,z}}{\gamma(1 + uv'_x/c^2)} \tag{49}$$

Relativitic Momentum - $m = \gamma m_0$, $\vec{p} = \gamma m_0 \vec{v}$

You can't go faster than light - Relativistic momentum $\to \infty$ as $v \to c$, so an infinite force is needed to accelerate a mass to speed c

Force at relativistic speeds - $\vec{F} = m\vec{a}$ leads to some weird stuff by working from $\vec{p} = \gamma m_0 \vec{v}$ like the force not having to be parallel to acceleration, so you mostly want to avoid these and work instead with momentum and energy, which are conserved.

5.3 Energy

Kinetic Energy - $E_K = (\gamma - 1)m_0c^2$

Mass/Energy Equivalence - Energy and Mass become indistinguishable at relativistic speeds. Assuming all mass can be converted to energy,

$$E = E_0 + E_K = m_0c^2 + (\gamma - 1)m_0c^2 = mc^2 \tag{50}$$

The Relation Equation - $E^2 - p^2c^2 = m_0^2c^4 = E_0^2$, this equation is also Lorentz invariant

Massless Particles - Using above as $m_0 \to 0$ and $v \to c$ yields $E = pc$

5.4 Odd Bits

Causality - Events are 'causal' if $\Delta x < c\Delta t$, i.e light has time to travel between the events (to transmit information) so the first *could* (not neccessarily!) have caused the second. Causailty is Lorentz invariant, so doesn't change depending on frame (you'll never have something happen in a different order than you observed)

Light Cones - The speed of light sets a limit on what events can affect each other, defining a 'past' and 'future', while allowing completely independant events in the past to be observed at the same time in the future.

Maximum Travel - Assuming no friction and constant acceleration, the distance measured from the place you started at's frame is given by

$$x(t') = \frac{c^2}{a}\left(\cosh\left(\frac{gt'}{c}\right) - 1\right) \tag{51}$$

An acceleration of $1g$ and a time in your frame t' of 10 years, get $x = 14{,}800$ light years.

5.5 History Lesson

Luminiferous Aether - A medium for light that needed to be super stiff but not transmit anything except light and be everywhere, was invented to fix Maxell's equation for the speed of disturbances in a magnetic field, $c = 1/\sqrt{\mu_0\epsilon_0}$ as their findings didn't fit with galilean velocity addition

Michelson-Morley Experiment - Light going against the 'Aether Wind' created by the Earth and Sun moving through space at great speeds, should have its speed reduced (/increased if it's going with the wind)
As light moves so quickly, the experiment compared the times to travel against/with and perpendicular to the wind measured at the same time, and then since it still needed to be too precise, compared the difference in the difference when the experiment is rotated 90 degrees (and then approximate with binomial). The experiment done with interferometry to get great travel distances, and on a mercury bath to reduce vibration and enable rotation, consistently found no differences, suggesting no 'Aether Drift' existed.

Lorentz-Fitzgerald Contraction - Suggesting a contraction factor of $1/\gamma$ (although not called that at the time) for objects moving along the aether wind would fix some problems, but more advanced (versions of the M-M) experiments (i.e $L1 \neq L2$) ruled this out too.